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Braids and partial permutations

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Abstract

We study the inverse braid monoid $\mathcal{IB}_n$ introduced by Easdown and Lavers in 2004. We completely describe the factorizable structure of $\mathcal{IB}_n$ and use this to give a new proof of the Easdown–Lavers presentation; we also derive several new presentations, each of which gives rise to a new presentation of the symmetric inverse monoid. We then define and study the pure inverse braid monoid $\mathcal{IP}_n$ which is related to $\mathcal{IB}_n$ in the same way that the pure braid group is related to the braid group.

MSC: primary 20F36; secondary 20M05, 20M18

Keywords: Braids; Braid monoids; Factorizable inverse monoids; Presentations

1. Introduction

The inverse braid monoid $\mathcal{IB}_n$, introduced by Easdown and Lavers in [10], is the monoid of all partial braids with at most n strings. The motivation for studying $\mathcal{IB}_n$ comes from several different directions. First, $\mathcal{IB}_n$ is a braid analogue of the symmetric inverse monoid $\mathcal{I}_n$ in the sense that there is an epimorphism $\mathcal{IB}_n \to \mathcal{I}_n$ which is a natural extension of the canonical projection from the braid group $\mathcal{B}_n$ to the symmetric group $\mathcal{S}_n$. (Braid analogues of other transformation semigroups have been studied in [11,14,15,18].) Secondly, geometric questions such as "*Given two braids β and γ, is it possible to remove a string from β and γ in order to produce equivalent* (*sub*)*braids?*" may be resolved algebraically, using a solution to the word problem

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in $\mathcal{IB}_n$; see [9,13]. Finally, it has recently been discovered [17] that $\mathcal{IB}_n$ arises naturally as the endomorphism monoid of an object in a certain "partialized category."

The authors of [10] showed that $\mathcal{IB}_n$ is an inverse monoid with semilattice of idempotents isomorphic to the power set of $\{1, \ldots, n\}$ considered as a semilattice under intersection. They also gave a presentation of $\mathcal{IB}_n$ which extends Artin's presentation [1] of $\mathcal{B}_n$, and then used this to deduce Popova's presentation [27] of $\mathcal{I}_n$. See also [16] where some of the results of [10] are reproduced using an entirely different approach. It was shown in [16] that $\mathcal{IB}_n$ is factorizable, a fact which is also implicit in certain calculations in [10]. A factorizable inverse monoid may be completely described by the following data: a semilattice E, a group G, an (anti-)action of G on E, and a collection G_e ($e \in E$) of subgroups of G; see [12,19]. In this article, we explore the factorizable structure of $\mathcal{IB}_n$, explicitly describing the data alluded to in the previous sentence. A crucial step is finding generating sets for the subgroups G_e. Using this structure, and a result from [12], we obtain a presentation of $\mathcal{IB}_n$, which will then be used to derive a number of alternative presentations including the original presentation [10], as well as several new and more symmetric presentations. We will also show how each presentation of $\mathcal{IB}_n$ gives rise to a presentation of $\mathcal{I}_n$. Two of these presentations include Moore's presentation [24] of the symmetric group along with a single extra generator and five additional relations, thus solving a problem posed in [28]. We then introduce the pure inverse braid monoid $\mathcal{IP}_n$ which is related to $\mathcal{IB}_n$ in the same way that the pure braid group $\mathcal{P}_n$ is related to $\mathcal{B}_n$. We conclude by giving a presentation of $\mathcal{IP}_n$ which includes Artin's presentation [2] of $\mathcal{P}_n$.

1.1. *Presentations*

Let X be a set, and denote by X^* the free monoid on X. Let $R \subseteq X^* \times X^*$ and write $R^\sharp$ for the congruence on X^* generated by R. We say that a monoid M has (*monoid*) *presentation* $\langle X \mid R \rangle$ if $M \cong X^*/R^\sharp$ or, equivalently, if there is an epimorphism $X^* \to M$ with kernel $R^\sharp$. If φ is such an epimorphism, then we say that M has presentation $\langle X \mid R \rangle$ *via* φ. Elements of X are called *generators*, and an element $(w_1, w_2) \in R$ is called a *relation* and will generally be displayed as an equation: $w_1 = w_2$. All presentations we consider will be monoid presentations.

1.2. *Factorizable inverse monoids*

Recall that an inverse monoid M is *factorizable* if $M = E_M G_M (= G_M E_M)$ where E_M and G_M denote the semilattice of idempotents and group of units of M respectively. See [7] for details regarding factorizable inverse monoids. We fix a factorizable inverse monoid M for the remainder of this section, and we write $E = E_M$ and $G = G_M$. For $e \in E$ and $g \in G$ define $e^g = geg^{-1}$. For $e \in E$ put

$$G_e = \{g \in G \mid eg = e\} = \{g \in G \mid ge = e\},$$

and choose a subset $S_e \subseteq G_e$ such that G_e is generated (as a subgroup) by S_e. Suppose now that E and G have presentations $\langle X_E \mid R_E \rangle$ and $\langle X_G \mid R_G \rangle$ via $\lambda : X_E^* \to E$ and $\mu : X_G^* \to G$ respectively. We also assume that X_E and X_G are disjoint, and that λ and μ are injective when restricted to X_E and X_G respectively. We also choose sets of words $\{\hat{e} \mid e \in E\} \subseteq X_E^*$ and $\{\hat{g} \mid g \in G\} \subseteq X_G^*$ such that $\hat{e}\lambda = e$ and $\hat{g}\mu = g$ for all $e \in E$ and $g \in G$. We make these choices

Fig. 1. The braids ς_i (left) and ς_i^{-1} (right) in $B = \mathcal{B}_n$.

so that $\widehat{x\lambda} = x$ and $\widehat{y\mu} = y$ for all $x \in X_E$ and $y \in X_G$. Now put $X_M = X_G \cup X_E$ and let $R_M \subseteq X_M^* \times X_M^*$ denote the set of relations $R_G \cup R_E \cup R_\rtimes \cup R_\sim$ where

$$R_\rtimes = \big\{\big(xy, \widehat{y\lambda^{x\mu}}x\big) \bigm| x \in X_G,\ y \in X_E\big\} \quad \text{and} \quad R_\sim = \big\{(\hat{e}\hat{g}, \hat{e}) \bigm| e \in E,\ g \in S_e\big\}.$$

Theorem 1. *(Easdown et al. [12]) With the notation of this section, the factorizable inverse monoid M has presentation $\langle X_M \mid R_M\rangle$ via*

$$X_M^* \to M : x \mapsto \begin{cases} x\lambda & \text{if } x \in X_E, \\ x\mu & \text{if } x \in X_G. \end{cases}$$

Remark 2. In [12] it was assumed that each G_e is generated as a sub*monoid* by S_e, but the proof may be modified trivially to give the result as stated here.

1.3. Braid groups

For the remainder of this article we fix a positive integer n, and we write $\mathbf{n} = \{1, \ldots, n\}$. The braid group $B = \mathcal{B}_n$ is the group of homotopy classes of braids on n strings [1]; for more information see [4] or [25]. Without causing confusion, we will always identify a braid with its homotopy class. For $1 \leqslant i \leqslant n-1$ we denote by ς_i and ς_i^{-1} the (homotopy classes of the) braids pictured in Fig. 1.

Let $X_B = \{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$ and denote by R_B the set of relations

$$\sigma_i^{\pm 1}\sigma_i^{\mp 1} = 1 \qquad \text{for all } i, \tag{F}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } |i-j| > 1, \tag{B1}$$

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \qquad \text{if } |i-j| = 1. \tag{B2}$$

Theorem 3. *(Artin [1]) The braid group $B = \mathcal{B}_n$ has presentation $\langle X_B \mid R_B\rangle$ via*

$$\varphi_B : X_B^* \to B : \sigma_i^{\pm 1} \mapsto \varsigma_i^{\pm 1}.$$

For $1 \leqslant i < j \leqslant n$ we define braids $\varsigma_{ij} = (\varsigma_{j-1}^{-1} \cdots \varsigma_{i+1}^{-1})\varsigma_i(\varsigma_{i+1} \cdots \varsigma_{j-1})$ and $\alpha_{ij} = \varsigma_{ij}^2$. These braids are pictured in Fig. 2. Notice in particular that $\varsigma_i = \varsigma_{i,i+1}$ and $\varsigma_i^2 = \alpha_{i,i+1}$ for all i.

Let $S = \mathcal{S}_n$ be the symmetric group on $\mathbf{n}$. For $\beta \in B$ we denote by $\bar{\beta} \in S$ the permutation induced by β so that the map $\beta \mapsto \bar{\beta}$ is the natural epimorphism $B \to S$. The kernel of this map is the pure braid group $P = \mathcal{P}_n = \{\beta \in B \mid \bar{\beta} = 1\}$. Referring to the diagram on the right-hand

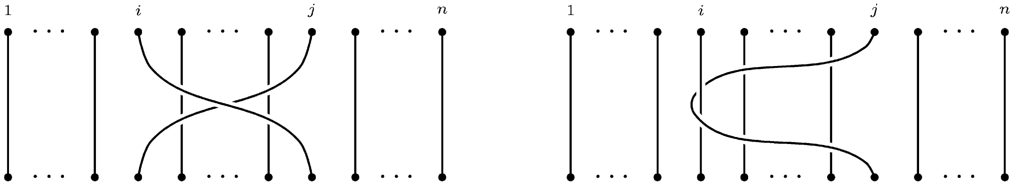

Fig. 2. The braids ς_{ij} (left) and α_{ij} (right) in $B = \mathcal{B}_n$.

side of Fig. 2, it is easy to see that $\alpha_{ij} \in P$ for all $1 \leqslant i < j \leqslant n$. Put $X_P = \{a_{ij}^{\pm 1} \mid 1 \leqslant i < j \leqslant n\}$ and let R_P denote the set of relations

$$a_{ij}^{\pm 1} a_{ij}^{\mp 1} = 1 \qquad \text{for all } i, j, \tag{F}$$

$$a_{rs}^{-1} a_{ij} a_{rs} = a_{ij} \qquad \text{if } i < r \text{ or } i > s, \tag{P1}$$

$$a_{rs}^{-1} a_{sj} a_{rs} = (a_{sj} a_{rj}) a_{sj} \big(a_{rj}^{-1} a_{sj}^{-1}\big), \tag{P2}$$

$$a_{rs}^{-1} a_{rj} a_{rs} = a_{sj} a_{rj} a_{sj}^{-1}, \tag{P3}$$

$$a_{rs}^{-1} a_{ij} a_{rs} = \big(a_{sj} a_{rj} a_{sj}^{-1} a_{rj}^{-1}\big) a_{ij} \big(a_{rj} a_{sj} a_{rj}^{-1} a_{sj}^{-1}\big) \quad \text{if } r < i < s, \tag{P4}$$

where in each of relations (P1)–(P4) we have $s < j$. There should be no confusion caused by choosing the label (F) to represent the free group relations in both R_B and R_P.

Theorem 4. *(Artin [2]) The pure braid group $P = \mathcal{P}_n$ has presentation $\langle X_P \mid R_P \rangle$ via*

$$\varphi_P : X_P^* \to P : a_{ij}^{\pm 1} \mapsto \alpha_{ij}^{\pm 1}.$$

For $2 \leqslant j \leqslant n$ let $\mathcal{U}_j$ be the subgroup of P generated by $\{\alpha_{1j}, \ldots, \alpha_{j-1,j}\}$.

Theorem 5. *(Artin [2]) We have the semidirect product decomposition*

$$\mathcal{P}_n = \mathcal{U}_n \rtimes \big(\mathcal{U}_{n-1} \rtimes \big(\cdots \rtimes (\mathcal{U}_3 \rtimes \mathcal{U}_2) \cdots\big)\big).$$

Although the previous two results are usually attributed to Artin [2], proofs may also be found in the papers [6] and [21] (respectively), which actually pre-date Artin's paper.

1.4. The symmetric inverse monoid

The symmetric inverse monoid $\mathcal{I}_n$ is the factorizable inverse monoid of all partial permutations on $\mathbf{n}$; for more information see [20] or [23]. We have

$$G_{\mathcal{I}_n} = \mathcal{S}_n \quad \text{and} \quad E_{\mathcal{I}_n} = \{\mathrm{id}_C \mid C \subseteq \mathbf{n}\}.$$

Since $\mathrm{id}_C \circ \mathrm{id}_D = \mathrm{id}_{C \cap D}$ for all $C, D \subseteq \mathbf{n}$ it is easy to see that $E_{\mathcal{I}_n}$ is isomorphic to the power set $E = P(\mathbf{n}) = \{C \mid C \subseteq \mathbf{n}\}$ considered as a semilattice under intersection. Put $X_E = \{\varepsilon_1, \ldots, \varepsilon_n\}$ and denote by R_E the set of relations

$$\varepsilon_i^2 = \varepsilon_i \quad \text{for all } i, \tag{E1}$$

$$\varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i \quad \text{for all } i, j. \tag{E2}$$

It is well known that E is a free semilattice on n generators (see for example [22]) and so we have the following. (Here and elsewhere, for $A \subseteq \mathbf{n}$, we write $A^c = \mathbf{n} \setminus A$.)

Theorem 6. *The powerset $E = (P(\mathbf{n}), \cap)$ has presentation $\langle X_E \mid R_E \rangle$ via*

$$\varphi_E : X_E^* \to E : \varepsilon_i \mapsto \{i\}^c.$$

2. The inverse braid monoid

A partial braid on n strings may be thought of as a (full) n-string braid with a number of strings removed. The inverse braid monoid $\mathcal{IB}_n$ is the monoid of all homotopy classes of partial braids on n strings [10]. Again, we will not distinguish between a partial braid and its homotopy class. The multiplication in $\mathcal{IB}_n$ is similar to that in the braid group $B = \mathcal{B}_n$; two partial braids are concatenated, and then any string fragments which do not connect the upper plane to the lower plane are removed. See Fig. 3 for an example.

A partial braid $\beta \in \mathcal{IB}_n$ induces a partial permutation $\bar{\beta}$ of $\mathbf{n}$ in a natural way. For example, the partial permutation associated to the partial braid in the top left diagram in Fig. 3 has the (partial) action $2 \mapsto 1$, $3 \mapsto 4$, $5 \mapsto 2$. It is clear that the map

$$\bar{\ }: \mathcal{IB}_n \to \mathcal{I}_n : \beta \mapsto \bar{\beta}$$

is a well-defined epimorphism whose restriction to B is the map $\bar{\ } : B \to S = \mathcal{S}_n$ discussed in Section 1.3.

Let $\beta \in B$ and $A \subseteq \mathbf{n}$. The *restriction* of β to A is the partial braid $\beta_A \in \mathcal{IB}_n$ obtained by removing all strings from β whose initial point is not indexed by an element of A.

Theorem 7. *The inverse braid monoid $\mathcal{IB}_n$ is a factorizable inverse monoid with*

$$E_{\mathcal{IB}_n} = \{1_A \mid A \subseteq \mathbf{n}\} \cong E = \big(P(\mathbf{n}), \cap\big) \quad \text{and} \quad G_{\mathcal{IB}_n} = B = \mathcal{B}_n.$$

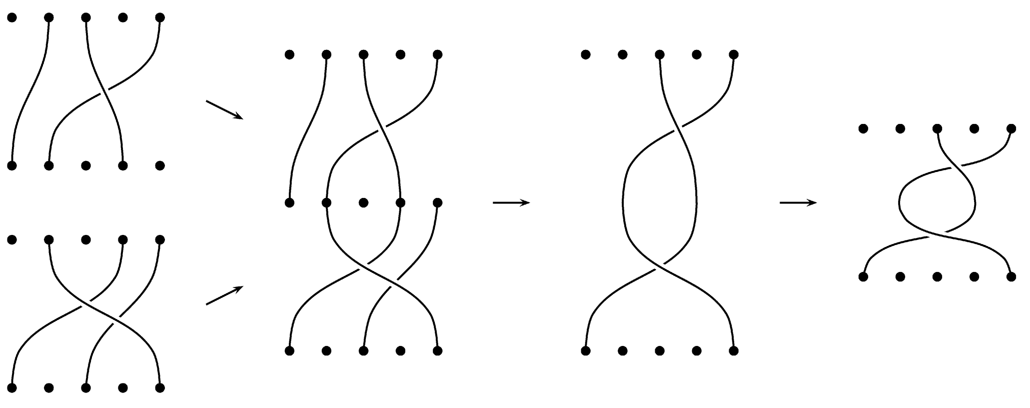

Fig. 3. Calculating a product in $\mathcal{IB}_5$.

Proof. It has already been shown that $\mathcal{IB}_n$ is inverse [10] and factorizable [16], and the statement concerning $E_{\mathcal{IB}_n}$ was proved in [10]. The fact that $G_{\mathcal{IB}_n} = B$ is clear in light of the fact that, writing $|\beta|$ for the number of strings in a partial braid β, we have $|\beta\gamma| \leqslant |\beta|$ for all $\beta, \gamma \in \mathcal{IB}_n$. $\square$

In order to utilize the factorizable structure of $\mathcal{IB}_n$, our next task is to describe the subgroups

$$B_{1_A} = \{\beta \in B \mid 1_A\beta = 1_A\} = \{\beta \in B \mid \beta_A = 1_A\}$$

which, for simplicity, we will denote by B_A. We first find a generating set for $B_A \cap P$. (Recall that $P = \mathcal{P}_n$ denotes the pure braid group.)

Lemma 8. *If $A \subseteq \mathbf{n}$, then $B_A \cap P$ is generated by the set*

$$\left\{\beta^{-1}\varsigma_i^2\beta \;\middle|\; i\bar{\beta} \in A^c \text{ or } (i+1)\bar{\beta} \in A^c\right\}.$$

Proof. Denote by X the set in the statement of the lemma. Clearly $X \subseteq P$. To see that $X \subseteq B_A$, we must show that $x_A = 1_A$ for any $x \in X$. With this in mind, let $x = \beta^{-1}\varsigma_i^2\beta$ where either $k = i\bar{\beta} \in A^c$ or $l = (i+1)\bar{\beta} \in A^c$. Replacing β by $\varsigma_i\beta$, if necessary, we may assume that $k < l$. Let h denote k if $k \in A^c$, or l otherwise. Referring to Fig. 4, it is clear that removing string h from x leaves (a partial braid which is homotopic to) $1_{\{h\}^c}$, and it follows that $x_A = 1_A$. Thus $\langle X\rangle \subseteq B_A \cap P$.

To show the reverse inclusion, suppose that $\beta \in B_A \cap P$. We first consider the case in which $A = \mathbf{r} = \{1, \ldots, r\}$ for some $0 \leqslant r \leqslant n$. By Theorem 5 we have $\beta = u_n \cdots u_2$ for (unique) braids $u_j \in \mathcal{U}_j$. Since $\beta_A = 1_A$ we see that $u_r = \cdots = u_2 = 1$. Thus $\beta = u_n \cdots u_{r+1}$ is a product of terms of the form $\alpha_{ij}^{\pm 1}$ with $j \in \{r+1, \ldots, n\} = A^c$. These elements are certainly in $X \cup X^{-1}$, showing that $\beta \in \langle X\rangle$.

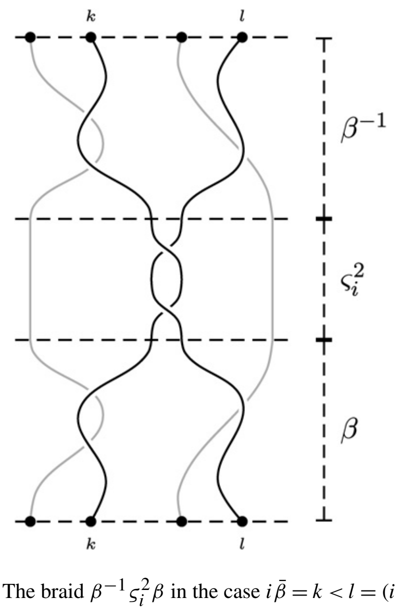

Fig. 4. The braid $\beta^{-1}\varsigma_i^2\beta$ in the case $i\bar{\beta} = k < l = (i+1)\bar{\beta}$.

To complete the proof, suppose now that $A \subseteq \mathbf{n}$ is arbitrary, and choose $\gamma \in B$ such that $A^\gamma = \{i \in \mathbf{n} \mid i\bar{\gamma} \in A\} = \mathbf{r}$ for some $0 \leqslant r \leqslant n$. Now since $\beta_A = 1_A$ we must have $(\gamma\beta\gamma^{-1})_\mathbf{r} = 1_\mathbf{r}$, so that $\gamma\beta\gamma^{-1} \in B_\mathbf{r} \cap P$. By the previous paragraph, $\gamma\beta\gamma^{-1}$ is a product of terms of the form $\alpha_{ij}^{\pm 1}$ with $j \in \mathbf{r}^c = (A^\gamma)^c$, from which it follows that β is a product of terms of the form $\gamma^{-1}\alpha_{ij}^{\pm 1}\gamma$ with $j \in (A^\gamma)^c$. Since $(A^\gamma)^c = (A^c)^\gamma$, we see that each of these terms is in $X \cup X^{-1}$, and the lemma is proved. $\square$

For the proof of the following result we denote by $t_{ij} = \bar{\varsigma}_{ij} \in S$ the transposition which interchanges the (distinct) elements $i, j \in \mathbf{n}$.

Corollary 9. *If $A \subseteq \mathbf{n}$, then B_A is generated by the set*

$$\left\{\varsigma_{ij} \;\middle|\; i, j \in A^c\right\} \cup \left\{\beta^{-1}\varsigma_i^2\beta \;\middle|\; i\bar{\beta} \in A^c \text{ or } (i+1)\bar{\beta} \in A^c\right\}.$$

Proof. Denote by Y the set in the statement of the corollary. Referring to the diagram on the left in Fig. 2 it is easy to see that $\varsigma_{ij} \in B_A$ if $i, j \in A^c$. Together with the first paragraph of the previous proof we see that $\langle Y \rangle \subseteq B_A$.

To show the reverse inclusion suppose that $\beta \in B_A$. Since $\beta_A = 1_A$ we see that $i\bar{\beta} = i$ for all $i \in A$. Thus we may write $\bar{\beta} = t_{i_1 j_1} \cdots t_{i_k j_k}$ for some $1 \leqslant i_s < j_s \leqslant n$ with $i_s, j_s \in A^c$ for each s. Put $\gamma = \varsigma_{i_1 j_1} \cdots \varsigma_{i_k j_k} \in \langle Y \rangle$. Then $\beta = (\beta\gamma^{-1})\gamma$ and we are done by the previous lemma since $\beta\gamma^{-1} \in B_A \cap P$. $\square$

3. Presentations of $\mathcal{IB}_n$

In this section we obtain several presentations of the inverse braid monoid $\mathcal{IB}_n$, each of which extends Artin's presentation [1] of the braid group $B = \mathcal{B}_n$. Since $\mathcal{IB}_n$ is a factorizable inverse monoid, we may apply Theorem 1 in order to obtain an initial presentation. We then remove many superfluous generators and relations to obtain the presentation discovered by Easdown and Lavers [10], and reproved in [16] using the theory of groupoids. This presentation may then be further simplified yielding a presentation which includes Artin's presentation of B along with a single extra generator and seven additional relations. We then explore a number of new presentations which involve larger sets of generators and relations, but display more of the natural symmetry possessed by $\mathcal{IB}_n$.

We first establish the notation which, in addition to that of Section 1, will be used throughout this section. Write $\sim_B = R_B^\sharp$ and choose a set of words $\{\hat{\beta} \mid \beta \in B\} \subseteq X_B^*$ such that $\hat{\beta}\varphi_B = \beta$ for all $\beta \in B$. We make these choices so that $\hat{\varsigma}_i = \sigma_i$ for all i, and

$$\hat{\varsigma}_{ij} = \left(\sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1}\right)\sigma_i(\sigma_{i+1} \cdots \sigma_{j-1}) \quad \text{and} \quad \hat{\alpha}_{ij} = \left(\sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1}\right)\sigma_i^2(\sigma_{i+1} \cdots \sigma_{j-1})$$

for all $1 \leqslant i < j \leqslant n$. If $w = \sigma_{i_1}^{\pm 1} \cdots \sigma_{i_k}^{\pm 1} \in X_B^*$, we write $w^{-1} = \sigma_{i_k}^{\mp 1} \cdots \sigma_{i_1}^{\mp 1}$, noting that $ww^{-1} \sim_B w^{-1}w \sim_B 1$. We will also assume that $\widehat{\beta^{-1}} = \hat{\beta}^{-1}$ for all $\beta \in B$. We also define a homomorphism $X_B^* \to S = \mathcal{S}_n$ by $w \mapsto \overline{w} = \overline{w\varphi_B}$.

For $i \in \mathbf{n}$ write $\xi_i = 1_{\{i\}^c} \in \mathcal{IB}_n$ for the partial braid obtained by removing the ith string from the identity braid. See Fig. 5 for an illustration. Note that if $A \subseteq \mathbf{n}$ and $A^c = \{i_1, \ldots, i_k\}$, then we have $1_A = \xi_{i_1} \cdots \xi_{i_k}$.

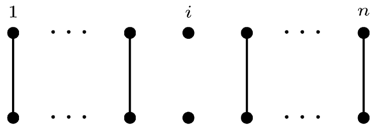

Fig. 5. The partial braid $\xi_i \in \mathcal{IB}_n$.

By Theorems 6 and 7, we see that $E_{\mathcal{IB}_n}$ has presentation $\langle X_E \mid R_E \rangle$ via

$$X_E^* \to E_{\mathcal{IB}_n} : \varepsilon_i \mapsto \xi_i.$$

Without causing confusion, and since we will no longer refer to the map $X_E^* \to (P(\mathbf{n}), \cap)$ defined in Theorem 6, we will denote this new map $X_E^* \to E_{\mathcal{IB}_n}$ by φ_E. We write $\sim_E = R_E^\sharp$, and choose a set of words $\{\hat{1}_A \mid A \subseteq \mathbf{n}\} \subseteq X_E^*$ such that $\hat{1}_A \varphi_E = 1_A$ for all $A \subseteq \mathbf{n}$. In particular, we have $\hat{1}_A \varepsilon_i \sim_E \hat{1}_A$ whenever $A \subseteq \mathbf{n}$ and $i \in A^c$.

3.1. An initial presentation

We now gather the remaining information needed in order to apply Theorem 1. For all i, r we have

$$\xi_i^{\varsigma_r^{\pm 1}} = \varsigma_r^{\pm 1} \xi_i \varsigma_r^{\mp 1} = \begin{cases} \xi_{i-1} & \text{if } r = i-1, \\ \xi_{i+1} & \text{if } r = i, \\ \xi_i & \text{otherwise.} \end{cases}$$

Alternatively, writing $s_r = t_{r,r+1} \in S$ for the simple transposition which interchanges r and $r+1$, we see that $\xi_i^{\varsigma_r^{\pm 1}} = \xi_{i s_r}$ for all i, r. Thus we may take $R_\rtimes$ to be the set of relations

$$\sigma_r^{\pm 1} \varepsilon_i = \varepsilon_{i s_r} \sigma_r^{\pm 1} \quad \text{for all } i, r. \tag{$\rtimes$}$$

Let $R_\sim$ be the set of relations

$$\hat{1}_A \hat{\varsigma}_{ij} = \hat{1}_A \quad \text{for all } A \subseteq \mathbf{n} \text{ and } i, j \in A^c, \tag{~ 1}$$

$$\hat{1}_A \hat{\beta}^{-1} \sigma_i^2 \hat{\beta} = \hat{1}_A \quad \text{for all } A \subseteq \mathbf{n} \text{ and } \beta \in B \text{ with } i\bar{\beta} \in A^c \text{ or } (i+1)\bar{\beta} \in A^c. \tag{~ 2}$$

Then by Theorem 1 and Corollary 9 we have the following.

Corollary 10. *The inverse braid monoid $\mathcal{IB}_n$ has presentation*

$$\langle X_B \cup X_E \mid R_B \cup R_E \cup R_\rtimes \cup R_\sim \rangle$$

via

$$\sigma_r^{\pm 1} \mapsto \varsigma_r^{\pm 1}, \quad \varepsilon_i \mapsto \xi_i.$$

3.2. The Easdown–Lavers presentation

Our task in this section is to reduce the number of generators and relations in the presentation of $\mathcal{IB}_n$ from Corollary 10 and thereby deduce (Theorem 21) the presentation originally discovered by Easdown and Lavers [10] (see also [16]). This presentation may then be simplified further (see Remark 23). We conclude the section by deriving a second presentation (see Theorem 24 and Remarks 25 and 26) which is similar in flavor to the Easdown–Lavers presentation.

Beginning with the presentation in Corollary 10, we rename $\varepsilon_1 = e$. Then by $R_{\rtimes}$ and (F) we see that the relations

$$\varepsilon_i = \sigma_{i-1}\cdots\sigma_1 e\sigma_1^{-1}\cdots\sigma_{i-1}^{-1} \quad \text{for all } i \tag{$*$}$$

are in $(R_B \cup R_E \cup R_{\rtimes} \cup R_{\sim})^{\sharp}$. Thus we may remove all generators ε_i with $i \neq 1$, replacing their every occurrence in the relations by the word on the right-hand side of $(*)$, which we denote by e_i. (Note in particular that $e_1 = e$.) We denote the resulting relations by (E1)$'$, (E2)$'$, $(\rtimes)'$, $(\sim 1)'$, and $(\sim 2)'$. The entire sets of relations which have been modified in this way will be denoted by R_E', $R_{\rtimes}'$, and $R_{\sim}'$. Let $X_{I\!B} = X_B \cup \{e\}$.

Corollary 11. *The inverse braid monoid $\mathcal{IB}_n$ has presentation*

$$\langle X_{I\!B} \mid R_B \cup R_E' \cup R_{\rtimes}' \cup R_{\sim}'\rangle$$

via

$$\sigma_i^{\pm 1} \mapsto \varsigma_i^{\pm 1}, \quad e \mapsto \xi_1.$$

For the duration of Section 3.2, we will write $\approx\, = (R_B \cup R_E' \cup R_{\rtimes}' \cup R_{\sim}')^{\sharp}$.

Lemma 12. *The following relations are in $\approx$:*

$$e^2 = e, \tag{IB1}$$

$$e\sigma_i = \sigma_i e \qquad \text{if } i \neq 1, \tag{IB2}$$

$$e\sigma_1 e\sigma_1 = \sigma_1 e\sigma_1 e = e\sigma_1 e, \tag{IB3}$$

$$e\sigma_1^2 = \sigma_1^2 e = e. \tag{IB4}$$

Proof. Now relation (IB1) is part of (E1)$'$, and (IB2) is part of $(\rtimes)'$. For (IB3) and (IB4) it suffices, by Corollary 11, to show that the equations $\xi\varsigma\xi\varsigma = \varsigma\xi\varsigma\xi = \xi\varsigma\xi$ and $\xi\varsigma^2 = \varsigma^2\xi = \xi$ hold in $\mathcal{IB}_n$ where $\varsigma = \varsigma_1$ and $\xi = \xi_1$. Figure 6 shows that the first equation holds. The second equation may be checked analogously. □

Let $R_{I\!B}$ denote the set of relations (F), (B1)–(B2), and (IB1)–(IB4), and denote by $\approxeq$ the congruence $R_{I\!B}^{\sharp}$. By Lemma 12 we see that $\approxeq\, = (R_{I\!B} \cup R_E' \cup R_{\rtimes}' \cup R_{\sim}')^{\sharp}$. Our aim is to show that $R_E' \cup R_{\rtimes}' \cup R_{\sim}' \subseteq\, \approxeq$ so that $\approx\, = \,\approxeq$ and $\mathcal{IB}_n$ has presentation $\langle X_{I\!B} \mid R_{I\!B}\rangle$.

Lemma 13. *If $w \in X_B^*$ and $w\varphi_B \in P = \mathcal{P}_n$ then $we \approxeq ew$.*

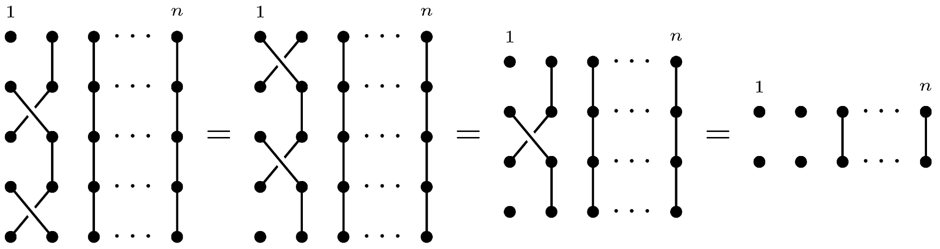

Fig. 6. Relation ($\mathbb{B}3$): $\xi\varsigma\xi\varsigma = \varsigma\xi\varsigma\xi = \xi\varsigma\xi$ where $\varsigma = \varsigma_1$ and $\xi = \xi_1$.

Proof. First note that for any $1 \leqslant i < j \leqslant n$, we have $\hat{\alpha}_{ij}^{\pm 1} e \approx e\hat{\alpha}_{ij}^{\pm 1}$ by ($\mathbb{B}2$), ($\mathbb{B}4$), and (F). The result now follows since $w \sim_B \hat{\alpha}_{i_1 j_1}^{\pm 1} \cdots \hat{\alpha}_{i_k j_k}^{\pm 1}$ for some i_s, j_s. $\Box$

For $i \in \mathbf{n}$ put $w_i = \sigma_{i-1} \cdots \sigma_1$, so that $e_i = w_i e w_i^{-1}$ by definition.

Corollary 14. *If $w \in X_B^*$, $w\varphi_B \in P$, and $i \in \mathbf{n}$, then $we_i \approx e_i w$.*

Proof. Now $(w_i^{-1} w w_i)\varphi_B = (w_i\varphi_B)^{-1}(w\varphi_B)(w_i\varphi_B) \in P$ and so, by (F) and Lemma 13, we have

$$we_i \approx w_i\big(w_i^{-1} w w_i\big)e w_i^{-1} \approx w_i e\big(w_i^{-1} w w_i\big)w_i^{-1} \approx e_i w,$$

completing the proof. $\Box$

Recall that $s_r \in S = \mathcal{S}_n$ denotes the simple transposition which interchanges r and $r+1$.

Lemma 15. *If $1 \leqslant r \leqslant n-1$, $i \in \mathbf{n}$, and $\ell \in \{\pm 1\}$, then $\sigma_r^{\ell} e_i \sigma_r^{-\ell} \approx e_{i s_r}$.*

Proof. First note that we have $\sigma_r e_i \sigma_r^{-1} \approx \sigma_r^{-1}\sigma_r^2 e_i \sigma_r^{-2}\sigma_r \approx \sigma_r^{-1} e_i \sigma_r$ by Corollary 14 and (F). Thus it suffices to prove the lemma for any choice of ℓ.

We consider the cases $r > i$, $r = i$, $r = i-1$, and $r < i-1$ separately. If $r > i$ then by (B1) we have $\sigma_r w_i \sim_B w_i \sigma_r$ from which it follows, by ($\mathbb{B}2$) and (F), that $\sigma_r e_i \sigma_r^{-1} \approx e_i$. If $r = i$ then $\sigma_i w_i = w_{i+1}$ and so $\sigma_i e_i \sigma_i^{-1} = e_{i+1}$. If $r = i-1$ then $\sigma_{i-1}^{-1} w_i \sim_B w_{i-1}$ by (F) and it follows that $\sigma_{i-1}^{-1} e_i \sigma_{i-1} \approx e_{i-1}$. If $r < i-1$ then $\sigma_r w_i \approx w_i \sigma_{r+1}$, as may easily be checked using the braid relations, so that $\sigma_r e_i \sigma_r^{-1} \approx e_i$ by ($\mathbb{B}2$) and (F). $\Box$

This shows that $R'_{\rtimes} \subseteq\ \approx$. The next result follows from Lemma 15 and a simple induction, noting that $s_r = \overline{\sigma_r^{\pm 1}}$ for all r.

Corollary 16. *If $i \in \mathbf{n}$ and $w \in X_B^*$, then $w^{-1} e_i w \approx e_{i\overline{w}}$.*

Lemma 17. *If $i \in \mathbf{n}$, then $e_i^2 \approx e_i$.*

Proof. This follows immediately from ($\mathbb{B}1$) and (F). $\Box$

Lemma 18. *If $i, j \in \mathbf{n}$, then $e_i e_j \approx e_j e_i$.*

Proof. If $i = j$ then there is nothing to show so suppose that $i \neq j$. First notice that by (F), (IB3), and (IB4) we have

$$e_1 e_2 = e\sigma_1 e\sigma_1^{-1} \approx e\sigma_1 e\sigma_1\sigma_1^{-2} \approx \sigma_1 e\sigma_1 e\sigma_1^{-2} \approx \sigma_1 e\sigma_1\sigma_1^{-2} e \approx \sigma_1 e\sigma_1^{-1} e = e_2 e_1.$$

Now choose a word $w \in X_B^*$ such that $i = 1\overline{w}$ and $j = 2\overline{w}$. Then by Corollary 16, (F), and the previous calculation, we have

$$e_i e_j = e_{1\overline{w}} e_{2\overline{w}} \approx w^{-1} e_1 w w^{-1} e_2 w \approx w^{-1} e_1 e_2 w \approx w^{-1} e_2 e_1 w \approx e_{2\overline{w}} e_{1\overline{w}} = e_j e_i,$$

and the proof is complete. □

Lemmas 17 and 18 show that $R'_E \subseteq \approx$.

Lemma 19. *If $A \subseteq \mathbf{n}$ and $i, j \in A^c$ with $i < j$, then $\hat{1}_A \hat{\varsigma}_{ij} \approx \hat{1}_A$.*

Proof. Observe first that by (F) and (IB3) we have

$$e_1 e_2 \sigma_1 = e\sigma_1 e\sigma_1^{-1}\sigma_1 \approx e\sigma_1 e\sigma_1\sigma_1^{-1} \approx e\sigma_1 e\sigma_1^{-1} = e_1 e_2.$$

Now put $u = (\sigma_2 \cdots \sigma_{j-1})(\sigma_1 \cdots \sigma_{i-1}) \in X_B^*$. It is easy to check diagrammatically that $\hat{\varsigma}_{ij} \sim_B u^{-1}\sigma_1 u$. Using this, together with the fact that $1\bar{u} = i$ and $2\bar{u} = j$, we see that

$$\begin{aligned}
\hat{1}_A \hat{\varsigma}_{ij} &\approx \hat{1}_A e_i e_j \hat{\varsigma}_{ij} && \text{by Lemmas 17 and 18}\\
&\approx \hat{1}_A \big(u^{-1} e_1 u\big)\big(u^{-1} e_2 u\big)\big(u^{-1}\sigma_1 u\big) && \text{by Corollary 16}\\
&\approx \hat{1}_A u^{-1} e_1 e_2 \sigma_1 u && \text{by (F)}\\
&\approx \hat{1}_A u^{-1} e_1 e_2 u && \text{by the observation}\\
&\approx \hat{1}_A e_i e_j && \text{by (F) and Corollary 16}\\
&\approx \hat{1}_A && \text{by Lemmas 17 and 18,}
\end{aligned}$$

and the proof is complete. □

Lemma 20. *If $A \subseteq \mathbf{n}$, $1 \leqslant i \leqslant n-1$, and $w \in X_B^*$ with either $i\overline{w} \in A^c$ or $(i+1)\overline{w} \in A^c$, then $\hat{1}_A w^{-1}\sigma_i^2 w \approx \hat{1}_A$.*

Proof. Put $v = (\sigma_2 \cdots \sigma_i)(\sigma_1 \cdots \sigma_{i-1}) \in X_B^*$. As in the previous proof (putting $j = i + 1$ in the word u), we have $\sigma_i = \hat{\varsigma}_{i,i+1} \sim_B v^{-1}\sigma_1 v$. Using this, together with the fact that $1\bar{v} = i$ and $2\bar{v} = i + 1$, we see that

$$\begin{aligned}
e_i \sigma_i^2 &\approx \big(v^{-1} e_1 v\big)\big(v^{-1}\sigma_1^2 v\big) && \text{by Corollary 16}\\
&\approx v^{-1} e_1 \sigma_1^2 v && \text{by (F)}\\
&\approx v^{-1} e_1 v && \text{by (IB4)}\\
&\approx e_i && \text{by Corollary 16,}
\end{aligned}$$

and

$$\begin{aligned}
e_{i+1}\sigma_i^2 &\approxeq \big(v^{-1}e_2v\big)\big(v^{-1}\sigma_1^2v\big) && \text{by Corollary 16}\\
&\approxeq v^{-1}e_2\sigma_1^2v && \text{by (F)}\\
&= v^{-1}\sigma_1e\sigma_1^{-1}\sigma_1^2v\\
&\approxeq v^{-1}\sigma_1e\sigma_1^{-1}v && \text{by (F) and } (\mathbb{B}4)\\
&= v^{-1}e_2v\\
&\approxeq e_{i+1} && \text{by Corollary 16.}
\end{aligned}$$

Now let h denote i if $i\overline{w} \in A^c$, or $i+1$ otherwise. We then have

$$\begin{aligned}
\hat{1}_Aw^{-1}\sigma_i^2w &\approxeq \hat{1}_Ae_{h\overline{w}}w^{-1}\sigma_i^2w && \text{by Lemmas 17 and 18}\\
&\approxeq \hat{1}_Aw^{-1}e_hww^{-1}\sigma_i^2w && \text{by Corollary 16}\\
&\approxeq \hat{1}_Aw^{-1}e_h\sigma_i^2w && \text{by (F)}\\
&\approxeq \hat{1}_Aw^{-1}e_hw && \text{by the previous two calculations}\\
&\approxeq \hat{1}_Ae_{h\overline{w}} && \text{by Corollary 16}\\
&\approxeq \hat{1}_A && \text{by Lemmas 17 and 18.}
\end{aligned}$$

This completes the proof. $\square$

Lemmas 19 and 20 show that $R'_{\sim} \subseteq \approxeq$. Together with the comments after the proofs of Lemmas 15 and 18, we see that $\approx = \approxeq$ and we have proved the following.

Theorem 21. *(Easdown and Lavers [10]) The inverse braid monoid $\mathcal{IB}_n$ has presentation $\langle X_{\mathbb{IB}} \mid R_{\mathbb{IB}}\rangle$ via*

$$\sigma_i^{\pm1} \mapsto \varsigma_i^{\pm1}, \quad e \mapsto \xi_1.$$

Remark 22. This presentation differs slightly from that given by the authors of [10]. There they map the generator e (called ε in [10]) to the partial braid ξ_n. Their relations may be obtained from ours (or ours from theirs) by changing each generator σ_i in $(\mathbb{IB}2)$–$(\mathbb{IB}4)$ to σ_{n-i}.

Remark 23. Relation $(\mathbb{IB}2)$ is equivalent to the assertion that the generator e commutes (in the quotient $X_{\mathbb{IB}}^*/R_{\mathbb{IB}}^\sharp$) with every word over $\{\sigma_2, \ldots, \sigma_{n-1}\}$. Let $w = \sigma_{n-1}\cdots\sigma_3$. It is easy to see diagrammatically that $\sigma_i \sim_B (w\sigma_2)^{-(i-2)}\sigma_2(w\sigma_2)^{i-2}$ for all $2 \leqslant i \leqslant n-1$; see also [1,8]. We see then that relation $(\mathbb{IB}2)$ may be replaced by the relations

$$e\sigma_2 = \sigma_2e \quad \text{and} \quad e\sigma_{n-1}\cdots\sigma_3 = \sigma_{n-1}\cdots\sigma_3e.$$

The resulting presentation of $\mathcal{IB}_n$ contains Artin's (monoid) presentation of B along with the additional generator e and seven extra relations.

We conclude this section with another presentation of $\mathcal{IB}_n$ which also includes Artin's presentation of B and a single extra generator. We begin with the presentation $\langle X_{\mathbb{IB}} \mid R_{\mathbb{IB}}\rangle$ of Theorem 21, and introduce a new generator f along with the relation

$$f = e\sigma_1.$$

Now by ($\mathbb{IB}$4) we have $(e\sigma_1)\sigma_1 \approx e$ and it follows that we may remove e as a generator, replacing its every occurrence in the relations by $f\sigma_1$. The modified relations are thus (F), (B1)–(B2), and

$$\begin{aligned} f\sigma_1 f\sigma_1 &= f\sigma_1, \\ f\sigma_1\sigma_i &= \sigma_i f\sigma_1 \qquad \text{if } i \neq 1, \\ f\sigma_1\sigma_1 f\sigma_1\sigma_1 = \sigma_1 f\sigma_1\sigma_1 f\sigma_1 &= f\sigma_1\sigma_1 f\sigma_1, \\ f\sigma_1\sigma_1^2 = \sigma_1^2 f\sigma_1 &= f\sigma_1. \end{aligned}$$

Using (F) we see that the first of these relations is equivalent to

$$f\sigma_1 f = f \tag{$\mathbb{IB}1)'$}$$

and, using (B1) and (F), that the second line of relations is equivalent to

$$f\sigma_1\sigma_2 = \sigma_2 f\sigma_1, \tag{$\mathbb{IB}2)'$}$$

$$f\sigma_i = \sigma_i f \qquad \text{if } i \geqslant 3. \tag{$\mathbb{IB}3)'$}$$

Then, using (F), we see that the fourth line of relations is equivalent to

$$f\sigma_1^2 = \sigma_1^2 f = f \tag{$\mathbb{IB}5)'$}$$

and, using (F) and ($\mathbb{IB}$5)$'$, that the third line of relations is equivalent to

$$f^2\sigma_1 = \sigma_1 f^2 = f^2. \tag{$\mathbb{IB}4)'$}$$

Let $X'_{\mathbb{IB}} = X_B \cup \{f\}$, and let $R'_{\mathbb{IB}}$ be the set of relations (F), (B1)–(B2), and ($\mathbb{IB}$1)–($\mathbb{IB}$5)$'$. We have shown the following.

Theorem 24. *The inverse braid monoid $\mathcal{IB}_n$ has presentation $\langle X'_{\mathbb{IB}} \mid R'_{\mathbb{IB}}\rangle$ via*

$$\sigma_i^{\pm 1} \mapsto \varsigma_i^{\pm 1}, \quad f \mapsto (\varsigma_1)_{\{1\}^c}.$$

Remark 25. If we made the substitution $g = \sigma_1 e$ instead of $f = e\sigma_1$ then we obtain a presentation of $\mathcal{IB}_n$ with generators $X_B \cup \{g\}$. The relations in this presentation are precisely those obtained by changing f to g in relations ($\mathbb{IB}$1)$'$ and ($\mathbb{IB}$3)–($\mathbb{IB}$5)$'$, and replacing relation ($\mathbb{IB}$2)$'$ by

$$\sigma_2\sigma_1 g = \sigma_1 g\sigma_2.$$

In this presentation, the generator g is sent to the partial braid $(\varsigma_1)_{\{2\}^c}$. The partial braids $(\varsigma_1)_{\{1\}^c}$ and $(\varsigma_1)_{\{2\}^c}$ are pictured in Fig. 7.

Fig. 7. The partial braids $(\varsigma_1)_{\{1\}^c}$ (left) and $(\varsigma_1)_{\{2\}^c}$ (right) in $\mathcal{IB}_n$.

Remark 26. As we have already stated, relations $(\mathbb{B}2)'$ and $(\mathbb{B}3)'$ are equivalent to

$$f\sigma_1\sigma_i = \sigma_i f\sigma_1 \quad \text{if } i \neq 1.$$

These relations are then equivalent to the assertion that $f\sigma_1$ commutes (in the quotient $(X'_{IB})^*/(R'_{IB})^\sharp$) with every word over $\{\sigma_2, \ldots, \sigma_{n-1}\}$. Thus, as in Remark 23, we may replace relations $(\mathbb{B}2)'$ and $(\mathbb{B}3)'$ by the two relations

$$f\sigma_1\sigma_2 = \sigma_2 f\sigma_1 \quad \text{and} \quad f\sigma_1\sigma_{n-1}\cdots\sigma_3 = \sigma_{n-1}\cdots\sigma_3 f\sigma_1,$$

resulting in a presentation which contains Artin's presentation of B along with one additional generator, f, and seven extra relations. Using (B1) and (F),we see that the longer of the above two relations may be replaced by

$$f\sigma_{n-1}\cdots\sigma_3 = \sigma_{n-1}\cdots\sigma_3 f.$$

3.3. More symmetric presentations

We now return to the presentation of $\mathcal{IB}_n$ given in Theorem 24. We will introduce a larger set of generators and relations in order to find a presentation which displays more of the symmetry possessed by $\mathcal{IB}_n$. With this in mind, we rename $f = f_1$ and introduce new generators $f_2, \ldots, f_{n-1}$ along with the relations

$$f_i = \left(\sigma_{i-1}^{-1}\cdots\sigma_1^{-1}\right)\left(\sigma_i^{-1}\cdots\sigma_2^{-1}\right) f_1(\sigma_2\cdots\sigma_i)(\sigma_1\cdots\sigma_{i-1}) \tag{D}$$

which define them in terms of the original generators. We will denote by F_i the word on the right-hand side of relation (D). For $1 \leqslant i \leqslant n-1$, let $\theta_i = (\varsigma_i)_{\{i\}^c}$ so that θ_i is the image of F_i under the map in the statement of Theorem 24. See Fig. 8 for an illustration. Now put $X''_{IB} = X_B \cup \{f_1, \ldots, f_{n-1}\}$, and denote by $\approx$ the congruence on $(X''_{IB})^*$ generated by R'_{IB} and (D).

Lemma 27. *The following relations are in* $\approx$:

$$\begin{aligned}
f_i\sigma_i f_i &= f_i && \textit{for all } i, && (\mathbb{B}1)'' \\
\sigma_i f_i &= f_{i+1}\sigma_{i+1} && \textit{for all } i \leqslant n-2, && (\mathbb{B}2)'' \\
\sigma_i f_j &= f_j\sigma_i && \textit{if } |i-j| > 1, && (\mathbb{B}3)'' \\
\sigma_i\sigma_j f_i &= f_j\sigma_i\sigma_j && \textit{if } |i-j| = 1, && (\mathbb{B}4)'' \\
\sigma_i f_i^2 = f_i^2 &= f_i^2\sigma_i && \textit{for all } i, && (\mathbb{B}5)'' \\
\sigma_i^2 f_i = f_i &= f_i\sigma_i^2 && \textit{for all } i. && (\mathbb{B}6)''
\end{aligned}$$

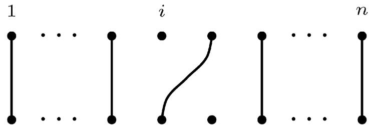

Fig. 8. The partial braid $\theta_i = (\varsigma_i)_{\{i\}^c}$ in $\mathcal{IB}_n$.

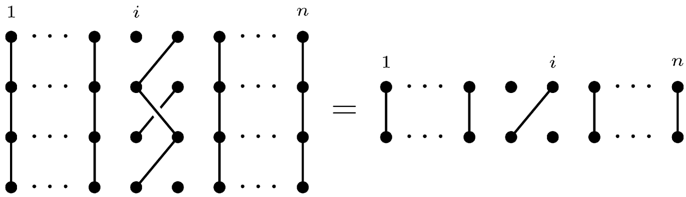

Fig. 9. Relation $(\mathbb{IB}1)''$: $\theta_i \varsigma_i \theta_i = \theta_i$ for all i.

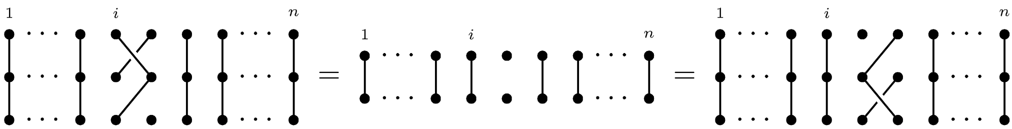

Fig. 10. Relation $(\mathbb{IB}2)''$: $\varsigma_i \theta_i = \theta_{i+1} \varsigma_{i+1}$ for all $i \leqslant n-2$.

Proof. By Theorem 24 it suffices to show that the relations hold as equations in $\mathcal{IB}_n$ when the generators σ_i and f_i are replaced by ς_i and θ_i respectively. We do this for relations $(\mathbb{IB}1)''$ and $(\mathbb{IB}2)''$ in Figs. 9 and 10. The other relations may be checked analogously. $\square$

By Lemma 27 we may add relations $(\mathbb{IB}1)''$–$(\mathbb{IB}6)''$ to the presentation. We may clearly remove relations $(\mathbb{IB}1)'$ and $(\mathbb{IB}3)$–$(\mathbb{IB}5)'$ since these relations are contained in various parts of $(\mathbb{IB}1)''$–$(\mathbb{IB}6)''$. Let $R''_{\mathbb{IB}}$ denote the set of relations (F), (B1)–(B2), and $(\mathbb{IB}1)''$–$(\mathbb{IB}6)''$.

Theorem 28. *The inverse braid monoid $\mathcal{IB}_n$ has presentation $\langle X''_{\mathbb{IB}} \mid R''_{\mathbb{IB}} \rangle$ via*

$$\sigma_i^{\pm 1} \mapsto \varsigma_i^{\pm 1}, \quad f_i \mapsto \theta_i.$$

Proof. For the duration of this proof we will denote by $\approx$ the congruence $(R''_{\mathbb{IB}})^\sharp$. It remains only to show that relations $(\mathbb{IB}2)'$ and (D) are in $\approx$. For $(\mathbb{IB}2)'$ we have

$$\begin{aligned} f_1\sigma_1\sigma_2 &\approx f_1\sigma_1\sigma_2\sigma_1\sigma_1^{-1} && \text{by (F)} \\ &\approx f_1\sigma_2\sigma_1\sigma_2\sigma_1^{-1} && \text{by (B2)} \\ &\approx \sigma_2\sigma_1 f_2\sigma_2\sigma_1^{-1} && \text{by } (\mathbb{IB}4)'' \\ &\approx \sigma_2\sigma_1\sigma_1 f_1\sigma_1^{-1} && \text{by } (\mathbb{IB}2)'' \\ &\approx \sigma_2 f_1\sigma_1\sigma_1\sigma_1^{-1} && \text{by } (\mathbb{IB}6)'' \\ &\approx \sigma_2 f_1\sigma_1 && \text{by (F).} \end{aligned}$$

To show that (D) is in $\approx$, we must show that $f_i \approx v_i^{-1} f_1 v_i$ for each i, where we have written $v_i = (\sigma_2 \cdots \sigma_i)(\sigma_1 \cdots \sigma_{i-1})$. Now if $i = 1$ then there is nothing to prove since $v_1 = 1$, so we assume that $2 \leqslant i \leqslant n-1$. By (B1) we have $v_i \sim_B v_{i-1}\sigma_i\sigma_{i-1}$. Using this, we have

$$
\begin{aligned}
v_i^{-1} f_1 v_i &\approx \sigma_{i-1}^{-1}\sigma_i^{-1} v_{i-1}^{-1} f_1 v_{i-1}\sigma_i\sigma_{i-1} && \\
&\approx \sigma_{i-1}^{-1}\sigma_i^{-1} f_{i-1}\sigma_i\sigma_{i-1} && \text{by an inductive hypothesis} \\
&\approx \sigma_{i-1}^{-1}\sigma_i^{-1}\sigma_i\sigma_{i-1} f_i && \text{by } (\mathbb{B}4)'' \\
&\approx f_i && \text{by (F),}
\end{aligned}
$$

completing the proof of the theorem. □

Remark 29.
Relations $(\mathbb{B}3)''$ and $(\mathbb{B}4)''$ form part of the *singular braid relations* (see [3] and [5]). It is natural then to ask whether the remaining singular braid relations

$$
\begin{aligned}
\sigma_i f_i = f_i \sigma_i & \quad \text{for all } i, \\
f_i f_j = f_j f_i & \quad \text{if } |i-j| > 1
\end{aligned}
$$

follow from relations $R''_{I\!B}$. In fact it is easy to check diagrammatically that the first of the missing singular braid relations does not hold, while the second does.

Remark 30. If instead we start with the presentation described in Remark 25, and introduce generators $g_1 = g$ and $g_2, \ldots, g_{n-1}$ where $g_i = v_i^{-1} g_1 v_i$ for each i, then we obtain the presentation $\langle X'''_{I\!B} \mid R'''_{I\!B} \rangle$, where $X'''_{I\!B} = X_B \cup \{g_1, \ldots, g_{n-1}\}$ and $R'''_{I\!B}$ is the set of relations obtained from $R''_{I\!B}$ by changing each f_i to g_i in $(\mathbb{B}1)''$ and $(\mathbb{B}3)''$–$(\mathbb{B}6)''$ and replacing $(\mathbb{B}2)''$ by

$$g_i \sigma_i = \sigma_{i+1} g_{i+1} \quad \text{for all } i \leqslant n-2.$$

In this presentation the generator g_i is mapped to the partial braid $(\varsigma_i)_{\{i+1\}^c}$.

4. Presentations of the symmetric inverse monoid

In [10] it was shown that a presentation of the symmetric inverse monoid $\mathcal{I}_n$ could be obtained from the presentation $\langle X_{I\!B} \mid R_{I\!B} \rangle$ by removing the generators $\sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}$ and replacing relation (F) by

$$\sigma_i^2 = 1 \quad \text{for all } i.$$

The presentation which results (after simplifying the relations) is attributed to Popova [27]; see [12,20,26] for a number of different proofs of Popova's presentation. In this section we show that all the presentations of the previous section yield presentations of $\mathcal{I}_n$.

To make the following result easier to state, we first establish some notation. Let X be any alphabet disjoint from X_B, and write $X_B^+ = \{\sigma_1, \ldots, \sigma_{n-1}\}$. We say that a word $w \in (X_B^+ \cup X)^*$ is *square-free* if w contains no subword from the set $\{\sigma_1^2, \ldots, \sigma_{n-1}^2\}$. Let $w \in (X_B \cup X)^*$. We define a square free word $\tilde{w} \in (X_B^+ \cup X)^*$ inductively as follows. Let w_0 be the word obtained from w by replacing any occurrence of a generator σ_i^{-1} by σ_i. If w_0 is square-free, put $\tilde{w} = w_0$. If w is not square-free, let $\tilde{w} = \tilde{w}_1$ where $w_1 \in (X_B^+ \cup X)^*$ is the word obtained from w_0 by deleting the left-most occurrence of a subword from $\{\sigma_1^2, \ldots, \sigma_{n-1}^2\}$.

Proposition 31. *Suppose that X is a set which is disjoint from X_B, and that the inverse braid monoid $\mathcal{IB}_n$ has presentation $\langle X_B \cup X \mid R_B \cup R \rangle$ via a map $\phi : (X_B \cup X)^* \to \mathcal{IB}_n$ under which $\sigma_i^{\pm 1}\phi = \varsigma_i^{\pm 1}$ for each i. Denote by R_O the set of relations*

$$\sigma_i^2 = 1 \quad \textit{for each } i, \tag{O}$$

and let $\widetilde{R}$ be the set of relations obtained from R by replacing every relation (w_1, w_2) by $(\tilde{w}_1, \tilde{w}_2)$, and deleting any such relation if $\tilde{w}_1$ and $\tilde{w}_2$ are identical words. Then the symmetric inverse monoid $\mathcal{I}_n$ has presentation $\langle X_B^+ \cup X \mid R_B^+ \cup R_O \cup \widetilde{R} \rangle$ via

$$x \mapsto \overline{x\phi},$$

where R_B^+ is the set of relations (B1)–(B2).

Proof. We first show that $\mathcal{I}_n$ has presentation $\langle X_B \cup X \mid R_B \cup R \cup R_O \rangle$ via

$$\phi' : (X_B \cup X)^* \to \mathcal{I}_n : x \mapsto \overline{x\phi}.$$

Put $\approx = (R_B \cup R)^\sharp$ and $\approxeq = (R_B \cup R \cup R_O)^\sharp$. Now ϕ' is an epimorphism since both ϕ and $\bar{\ }$ are epimorphisms. Since $s_i^2 = 1$ for each i, we see that $\approxeq \subseteq \ker \phi'$. It remains to show the reverse inclusion, so suppose that $(w_1, w_2) \in \ker \phi'$, and put $\beta_1 = w_1\phi$ and $\beta_2 = w_2\phi$. Then

$$\bar{\beta}_1 = w_1\phi' = w_2\phi' = \bar{\beta}_2.$$

Let $A = \mathrm{dom}(\bar{\beta}_1)$. We then have

$$\beta_1 = (\gamma_1)_A = 1_A\gamma_1 \quad \text{and} \quad \beta_2 = (\gamma_2)_A = 1_A\gamma_2$$

for some $\gamma_1, \gamma_2 \in B = \mathcal{B}_n = G_{\mathcal{IB}_n}$. Choose $u_1, u_2 \in X_B^*$ such that

$$u_1\phi = \gamma_1 \quad \text{and} \quad u_2\phi = \gamma_2,$$

and choose $w_A \in (X_B \cup X)^*$ such that $w_A\phi = 1_A$. We then have

$$w_1 \approx w_A u_1 \quad \text{and} \quad w_2 \approx w_A u_2$$

by assumption. Put $\pi = (u_1 u_2^{-1})\phi' \in S = \mathcal{S}_n$. Now for any $i \in A$ we have

$$i(u_1\phi') = i(w_1\phi') = i(w_2\phi') = i(u_2\phi').$$

Thus $i\pi = i$, whence $i\pi^{-1} = i$, for each $i \in A$. Thus we may write

$$\pi^{-1} = t_{p_1q_1} \cdots t_{p_kq_k}$$

for some $p_j, q_j \in A^c$. By Corollary 9 we have $\varsigma_{p_1q_1}\cdots\varsigma_{p_kq_k} \in B_A$, and it then follows that

$$1_A = 1_A\varsigma_{p_1q_1}\cdots\varsigma_{p_kq_k}$$

in $\mathcal{IB}_n$. Thus, by hypothesis, we have

$$w_A \approx w_A\hat{\varsigma}_{p_1q_1}\cdots\hat{\varsigma}_{p_kq_k}.$$

Now $(\hat{\varsigma}_{p_1q_1}\cdots\hat{\varsigma}_{p_kq_k})\phi' = \pi^{-1} = ((u_1u_2^{-1})\phi')^{-1}$ and so $(\hat{\varsigma}_{p_1q_1}\cdots\hat{\varsigma}_{p_kq_k}u_1u_2^{-1})\varphi_B \in P = \mathcal{P}_n$. Thus

$$\hat{\varsigma}_{p_1q_1}\cdots\hat{\varsigma}_{p_kq_k}u_1u_2^{-1} \sim_B \hat{\alpha}_{u_1v_1}^{\pm1}\cdots\hat{\alpha}_{u_\ell v_\ell}^{\pm1}$$

for some u_j, v_j. By (O) and (F) we have

$$\hat{\alpha}_{ij}^{\pm1} = \left(\sigma_{j-1}^{-1}\cdots\sigma_{i+1}^{-1}\right)\sigma_i^{\pm2}(\sigma_{i+1}\cdots\sigma_{j-1}) \approx 1$$

for all $1 \leqslant i < j \leqslant n$, so that

$$\begin{aligned} w_1 &\approx w_Au_1 \approx (w_Au_1u_2^{-1})u_2 \approx \left(w_A\hat{\varsigma}_{p_1q_1}\cdots\hat{\varsigma}_{p_kq_k}u_1u_2^{-1}\right)u_2 \\ &\approx \left(w_A\hat{\alpha}_{u_1v_1}^{\pm1}\cdots\hat{\alpha}_{u_\ell v_\ell}^{\pm1}\right)u_2 \approx w_Au_2 \approx w_2. \end{aligned}$$

This completes the proof that $\mathcal{I}_n$ has presentation $\langle X_B \cup X \mid R_B \cup R \cup R_O\rangle$ via ϕ'.

It now suffices to show that the presentation we have arrived at is equivalent to the presentation in the statement of the proposition. Now by (F) and (O) we have

$$\sigma_i \approx \sigma_i\sigma_i\sigma_i^{-1} \approx \sigma_i^{-1}$$

for all i. Thus we may remove all generators σ_i^{-1}, replacing their every occurrence in the remaining relations by σ_i. Relations (F) become duplicates of (O) and, as such, may be removed. Finally, observe that we may use relations (O) to transform any word $w \in (X_B^+ \cup X)^*$ into $\tilde{w}$, thus allowing us to replace R by $\widetilde{R}$. This completes the proof. □

As our first application of Proposition 31, we derive Popova's presentation [27] of $\mathcal{I}_n$. Put $X_I = \{\sigma_1, \ldots, \sigma_{n-1}, e\} = X_B^+ \cup \{e\}$, and let R_I be the set of relations (B1)–(B2), (O), and ($\mathbb{I}$B1)–($\mathbb{I}$B3). By Theorem 21 and Proposition 31 we have the following.

Theorem 32. *(Popova [27]) The symmetric inverse monoid $\mathcal{I}_n$ has presentation $\langle X_I \mid R_I\rangle$ via*

$$\sigma_i \mapsto s_i, \quad e \mapsto \mathrm{id}_{\{1\}^c}.$$

Remark 33. Again, this presentation differs from that proved in [10]; there the generator e was mapped to the partial permutation $\mathrm{id}_{\{n\}^c}$, and the relations were slightly different.

Remark 34. Applying Proposition 31 to the presentation described in Remark 23, followed by a simple manipulation of the resulting relations, yields a presentation of $\mathcal{I}_n$ with generators $X_B^+ \cup \{e\}$ and relations (B1)–(B2), (O), and

$$e^2 = e = \sigma_2 e \sigma_2 = \sigma_3 \cdots \sigma_{n-1} e \sigma_{n-1} \cdots \sigma_3 \quad \text{and} \quad e\sigma_1 e \sigma_1 = \sigma_1 e \sigma_1 e = e\sigma_1 e.$$

This solves one half of Open Problem 4 of [28]. The other half of the problem asks "given the generators $\sigma_1, \ldots, \sigma_{n-1}, e$, what is the minimum number of relations which may be added to (B1)–(B2) and (O) to yield a presentation of $\mathcal{I}_n$?" The author believes that the answer is five.

We may also use Proposition 31 to derive a presentation of $\mathcal{I}_n$ similar to the presentation of $\mathcal{IB}_n$ in Theorem 24. Let $X_I' = \{\sigma_1, \ldots, \sigma_{n-1}, f\} = X_B^+ \cup \{f\}$, and let R_I' be the set of relations (B1)–(B2), (O), and $(\mathbb{B}1)'$–$(\mathbb{B}4)'$.

Theorem 35. *The symmetric inverse monoid $\mathcal{I}_n$ has presentation $\langle X_I' \mid R_I' \rangle$ via*

$$\sigma_i \mapsto s_i, \quad f \mapsto s_1|_{\{1\}^c}.$$

Now put $X_I'' = X_B^+ \cup \{f_1, \ldots, f_{n-1}\}$ let R_I'' denote the set of relations (B1)–(B2), (O), and $(\mathbb{B}1)''$–$(\mathbb{B}5)''$. The next result follows from Theorem 28 and Proposition 31.

Theorem 36. *The symmetric inverse monoid $\mathcal{I}_n$ has presentation $\langle X_I'' \mid R_I'' \rangle$ via*

$$\sigma_i \mapsto s_i, \quad f_i \mapsto s_i|_{\{i\}^c}.$$

Remark 37. We may also deduce presentations of $\mathcal{I}_n$ from Remarks 25 and 30.

5. The pure inverse braid monoid

In this final section we introduce and study the pure inverse braid monoid. This monoid, denoted $\mathcal{IP}_n$, is related to the inverse braid monoid $\mathcal{IB}_n$ in exactly the same way that the pure braid group $P = \mathcal{P}_n$ is related to the braid group $B = \mathcal{B}_n$.

Analogously to the notion of pure braids, we say that a partial braid $\beta \in \mathcal{IB}_n$ is *pure* if $i\bar{\beta} = i$ for all $i \in \operatorname{dom}(\bar{\beta})$. The set of all pure partial braids is clearly a submonoid of $\mathcal{IB}_n$, being the preimage of $E_{\mathcal{I}_n}$ under the map $\bar{\ }: \mathcal{IB}_n \to \mathcal{I}_n$. We denote this submonoid, the *pure inverse braid monoid*, by $\mathcal{IP}_n$.

Let $A \subseteq \mathbf{n}$ and put

$$\mathcal{IP}_n^A = \{\beta \in \mathcal{IP}_n \mid \operatorname{dom}(\bar{\beta}) = A\} = \{\gamma_A \mid \gamma \in P\}.$$

Since $\mathcal{IP}_n^A$ is the preimage of the idempotent id_A under the $\bar{\ }$ map, it is therefore a subsemigroup of $\mathcal{IP}_n$. In fact, it is easy to check that $\mathcal{IP}_n^A$ is a sub*group* of $\mathcal{IP}_n$ and is isomorphic to $\mathcal{P}_k$, the pure braid group on $k = |A|$ strings. It is also clear that we have the disjoint union

$$\mathcal{IP}_n = \bigsqcup_{A \subseteq \mathbf{n}} \mathcal{IP}_n^A$$

so that the subgroups $\mathcal{IP}_n^A$ are in fact the *maximal* subgroups of $\mathcal{IP}_n$. The proof of the next result is routine.

Theorem 38. *The pure inverse braid monoid $\mathcal{IP}_n$ is a factorizable inverse monoid with*

$$E_{\mathcal{IP}_n} = E_{\mathcal{IB}_n} = \{1_A \mid A \subseteq \mathbf{n}\} \cong \big(P(\mathbf{n}), \cap\big) \quad \text{and} \quad G_{\mathcal{IP}_n} = P = \mathcal{P}_n.$$

Further, $1_A\beta = \beta 1_A$ for all $A \subseteq \mathbf{n}$ and $\beta \in P$, so that $\mathcal{IP}_n$ is a semilattice of (pure braid) groups.

We now work towards the derivation of a presentation of $\mathcal{IP}_n$. We first obtain an auxiliary result (Corollary 40) concerning the subgroups B_A defined in Section 2. In the statement of the next result, which was proved in [15], we use the symmetric notation $\alpha_{ji} = \alpha_{ij}$.

Lemma 39. *If $1 \leqslant i < j \leqslant n$ and $\beta \in B$, then there exists $\delta \in P$ such that*

$$\beta^{-1}\alpha_{ij}\beta = \delta^{-1}\alpha_{i\bar\beta, j\bar\beta}\delta.$$

The next result follows immediately from Lemmas 8 and 39, noting that $\varsigma_i^2 = \alpha_{i,i+1}$ for all i.

Corollary 40. *If $A \subseteq \mathbf{n}$, then $B_A \cap P$ is generated by the set*

$$\big\{\beta^{-1}\alpha_{ij}\beta \;\big|\; \beta \in P \text{ and } i \in A^c \text{ or } j \in A^c\big\}.$$

Recall, by Theorems 4 and 6, that P and $E_{\mathcal{IP}_n} = E_{\mathcal{IB}_n}$ have presentations $\langle X_P \mid R_P\rangle$ and $\langle X_E \mid R_E\rangle$ via φ_P and φ_E respectively. We choose sets of words $\{\hat\beta \mid \beta \in P\} \subseteq X_P^*$ and $\{\hat 1_A \mid A \subseteq \mathbf{n}\} \subseteq X_E^*$ such that $\hat\beta\varphi_P = \beta$ and $\hat 1_A\varphi_E = 1_A$ for all $\beta \in P$ and $A \subseteq \mathbf{n}$. It is important to note that our definition of $\hat\beta$ here (as a word over X_P) does not coincide with our earlier definition (as a word over X_B).

Now put $X_{IP} = X_P \cup X_E$, and let R_{IP} denote the set of relations $R_P \cup R_E$ together with

$$\varepsilon_i a_{rs} = a_{rs}\varepsilon_i \qquad \text{for all } i, r, s, \tag{IP1}$$

$$\varepsilon_i a_{ij} = \varepsilon_i \qquad \text{for all } i, j, \tag{IP2}$$

$$\varepsilon_j a_{ij} = \varepsilon_j \qquad \text{for all } i, j. \tag{IP3}$$

Theorem 41. *The pure inverse braid monoid $\mathcal{IP}_n$ has presentation $\langle X_{IP} \mid R_{IP}\rangle$ via*

$$a_{rs} \mapsto \alpha_{rs}, \quad \varepsilon_i \mapsto \xi_i.$$

Proof. Let φ denote the map in the statement of the theorem, and write $\approx\, = R_{IP}^\sharp$. To show that φ is surjective, suppose that $\beta \in \mathcal{IP}_n$. Then $\beta = \gamma_A$ for some $A \subseteq \mathbf{n}$ and $\gamma \in P$. But then $\beta = 1_A\gamma = (\hat 1_A\hat\gamma)\varphi$. Next, it is easy to check diagrammatically that $R_{IP} \subseteq \ker(\varphi)$. It remains only to show that $\ker(\varphi) \subseteq \approx$, so suppose that $(w_1, w_2) \in \ker(\varphi)$. Now by (IP1) and $R_P \cup R_E$ we have

$$w_1 \approx \hat 1_{A_1}\hat\beta_1 \quad \text{and} \quad w_2 \approx \hat 1_{A_2}\hat\beta_2$$

for some $A_1, A_2 \subseteq \mathbf{n}$ and $\beta_1, \beta_2 \in P$. We then have

$$(\beta_1)_{A_1} = 1_{A_1}\beta_1 = w_1\varphi = w_2\varphi = 1_{A_2}\beta_2 = (\beta_2)_{A_2},$$

from which it follows that $A_1 = A_2$ and $\beta_1\beta_2^{-1} \in B_{A_1}$. By Corollary 40, we have

$$\beta_1\beta_2^{-1} = \left(\gamma_1^{-1}\alpha_{i_1 j_1}^{\pm 1}\gamma_1\right)\cdots\left(\gamma_s^{-1}\alpha_{i_s j_s}^{\pm 1}\gamma_s\right)$$

for some $\gamma_1, \dots, \gamma_s \in P$ and $i_1, \dots, i_s, j_1, \dots, j_s \in \mathbf{n}$ with $i_t \in A_1^c$ or $j_t \in A_1^c$ for each t. For each t, let h_t denote i_t if $i_t \in A_1^c$, or j_t otherwise. We then have

$$\begin{aligned}
w_1 &\approx \hat{1}_{A_1}\hat{\beta}_1 \\
&\approx \hat{1}_{A_1}\hat{\beta}_1\hat{\beta}_2^{-1}\hat{\beta}_2 && \text{by (F)} \\
&\approx \hat{1}_{A_1}\varepsilon_{h_1}\cdots\varepsilon_{h_s}\left(\hat{\gamma}_1^{-1}a_{i_1 j_1}^{\pm 1}\hat{\gamma}_1\right)\cdots\left(\hat{\gamma}_s^{-1}a_{i_s j_s}^{\pm 1}\hat{\gamma}_s\right)\hat{\beta}_2 && \text{by } R_E \cup R_P \\
&\approx \hat{1}_{A_1}\left(\hat{\gamma}_1^{-1}\varepsilon_{h_1}a_{i_1 j_1}^{\pm 1}\hat{\gamma}_1\right)\cdots\left(\hat{\gamma}_s^{-1}\varepsilon_{h_s}a_{i_s j_s}^{\pm 1}\hat{\gamma}_s\right)\hat{\beta}_2 && \text{by (IP1) and (F)} \\
&\approx \hat{1}_{A_1}\left(\hat{\gamma}_1^{-1}\varepsilon_{h_1}\hat{\gamma}_1\right)\cdots\left(\hat{\gamma}_s^{-1}\varepsilon_{h_s}\hat{\gamma}_s\right)\hat{\beta}_2 && \text{by (IP2), (IP3), and (F)} \\
&\approx \hat{1}_{A_1}\varepsilon_{h_1}\cdots\varepsilon_{h_s}\hat{\beta}_2 && \text{by (IP1) and (F)} \\
&\approx \hat{1}_{A_1}\hat{\beta}_2 && \text{by } R_E \\
&= \hat{1}_{A_2}\hat{\beta}_2 \\
&\approx w_2,
\end{aligned}$$

and the proof is complete. □

Remark 42. The previous theorem may be proved in a similar fashion to Theorem 21; that is, by applying Theorems 1 and 38 and Corollary 40 to obtain an initial presentation which may then be simplified. However, the direct proof above is to be preferred since it is certainly quicker and more straightforward.

Remark 43. Unlike the situation with $\mathcal{IB}_n$, since the action of a pure braid on ξ_i is trivial, there is no hope of removing any of the generators ε_i in the above presentation of $\mathcal{IP}_n$.

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